LIE Ideals and Generalized Derivations on Prime Rings

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ABSTRACT

Let R be an associative ring. An additive mapping $F : R \rightarrow R$ is called generalized derivations if there exists a derivation $d : R \rightarrow R$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. The objective of the present paper is to extend the results for nonzero Lie ideal of a prime ring R.

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1. INTRODUCTION

Throughout the discussion, unless otherwise mentioned, R will denote an associative ring having at least two elements with center $Z(R)$. However, R may not have unity. For any $x,y \in R$, the symbol $[x,y]$ (resp. $(x \circ y)$) will denote the commutator $xy - yx$. For any $x; y \in R$; the symbol $[x; y]$ will denote the commutator $xy-yx$. The least positive integer n such that $nx = 0$ for all $x \in R$ is called the characteristic of the ring R and generally expressed as char (R). If no such positive integer exists, then R is said to have characteristic 0. Obviously, if char(R) $\neq$ m, then $mx = 0$ for some $x \in R$. Recall that a ring R is said to be prime if for any $a, b \in R$; $a R b = (0)$ implies $a = 0$ or $b = 0$. An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$ for all $u \in U$ and $r \in R$.

An additive mapping $d : R \rightarrow R$ is said to be a derivation of R if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. A derivation d is said to be inner if there exists $a \in R$ such that $d(x) = ax - xa$ for all $x \in R$: Following Brešar[3], an additive mapping $F : R \rightarrow R$ is said to be the generalized derivation of R with associated derivation d if $F(xy) = F(x)y + xd(y)$ for all $x; y \in R$. It is clear that the concept of generalized derivation covers both the concept of derivation and the concept of left multiplier (i.e., an additive mapping $T : R \rightarrow R$ satisfying $T(xy) = T(x)y$ for all $x; y \in R$). Clearly, every generalized derivation is a derivation but the converse is not true in general. Very recently, Ali et al. [1] established that a semiprime ring R with a non-zero ideal I must be commutative if it admits a generalized derivation F satisfying the properties $F(x)F(y) \pm xy \in Z(R)$ and $F(x)F(y) \pm yx \in Z(R)$ for all $x, y \in I$. In this line of investigation, we study the above mentioned identities in the setting of square closed Lie ideal of a prime ring.

2. PRELIMINARIES RESULTS

Before stating our main result, let us list some basic facts, which we will be used in the following:

Lemma 2.1. [2, Lemma 4] Let R be a prime ring with characteristic not two, $a, b \in R$. If U a non central Lie ideal of R and $aUb = (0)$, then $a = 0$ or $b = 0$.

Lemma 2.2. [4, Theorem 7] Let R be a prime ring with characteristic not two and U be a nonzero Lie ideal of R. If d is a nonzero derivation of R such that $[u, d(u)] \in Z(R)$, then $U \subsetneq Z(R)$.

3. MAIN RESULTS

Theorem 3.1. Let R be a prime ring such that char(R) $\neq 2$ and U be a Lie ideal of R such that $u^2 \in U$, for all $u \in U$. If there exists an additive mapping $F : R \rightarrow R$ associated with a derivation d of R such that $F(xy) = F(x)y + xd(y)$ for all x, y ∈ R. If $F(u)F(v) \pm uv \in Z(R)$ for all $u, v \in U$. Then $U \subsetneq Z(R)$.

Proof. First we consider the case

(3.1) $F(u)F(v) - uv \in Z(R)$ for all $u, v \in U$:

Replacing v with $2vw$, $w \in U$, we have

(3.2) $2(F(u)F(vw) - u(vw)) \in Z(R)$

which gives

(3.3) $2(F(u)(F(v)w + vd(w)) - uvw) \in Z(R)$ for all $u, v, w \in U$:

This can be further written as

(3.4) $2([F(u)(F(v)w + vd(w)) - uvw, w]) = 0$ for all $u; v, w \in U$:

Since char$(R) \neq 2$, we get

(3.5) $[F(u)(F(v)w + vd(w)) - uvw, w] = 0$ for all $u, v, w \in U$.

Application of (3.1) yields that

(3.6) $[F(u)vd(w),w] = 0$ for all $u, v, w \in U$.

Putting $v = wv$ in above relation and using the fact that char$(R) \neq 2$, we obtain

(3.7) $[F(u)wvd(w), w] = 0$ for all $u; v, w \in U$.

Now putting $u = 2uw$ in (3:6) and using the fact that char$(R) \neq 2$, we get

(3.8) $[F(u)wvd(w), w] + [ud(w)vd(w), w] = 0$ for all $u; v, w \in U$.

In view of (3:7), above relation reduces to

(3.9) $[ud(w)vd(w), w] = 0$ for all $u; v. w \in U$.

Substituting $2d(w)u$ for u in (3:9) and using it, we obtain

(3.10) $2([d(w), w]ud(w)vd(w)) = 0$ for all $u, v, w \in U$.

Since char$(R) \neq 2$, we get

(3.11) $[d(w), w]ud(w)vd(w) = 0$ for all $u, v, w \in U$.

This implies $[d(w), w]u[d(w), w]v[d(w), w] = 0$ that is, $[d(w), w]u[d(w),w]u[d(w),w] = (0)$ for all $w \in U$. Application of Lemma 2.1 yields that

(3.12) $[d(w),w] = 0$ for all $w \in U$.

In view of Lemma 2.2, $U \subseteq Z(R)$.

By the same argument, we obtain the same conclusion in case $F(u)F(v) + uv \in Z(R)$ for all $u, v \in U$. This completes the proof of theorem.

Theorem 3.2. Let R be a prime ring such that char$(R) \neq 2$ and U be a Lie ideal of R such that $u^2 \in U$, for all $u \in U$. If there exists an additive mapping $F : R \rightarrow R$ associated with a derivation d of R such that

$$F(xy) = F(x)y + xd(y) \text{ for all } x, y \in R.$$

If $F(u)F(v) \pm vu \in Z(R)$ for all $u, v \in U$. Then $U \subseteq Z(R)$.

Proof. We begin with the situation

(3.13) $F(u)F(v) - vu \in Z(R)$ for all $u, v \in U$.

Replacing v by $2vw$, we get

(3.14) $2(F(u)(F(v)w + vd(w)) - vwu) \in Z(R)$ for all $u, v, w \in U$.

This implies that

(3.15) $2((F(u)F(v) - vu)w + v[u, w] + F(u)vd(w)) \in Z(R)$ for all $u, v, w \in U$.

Application of (3:13) yields that

(3.16) $2([v[u, w], w] + [F(u)vd(w);w]) = 0$ for all $u, v, w \in U$.

Since char$(R) \neq 2$, we get

(3.17) $[v[u, w],w] + [F(u)vd(w), w] = 0$ for all $u, v, w \in U$.

Substituting $2uw$ for u in (3:16) and using the fact that char$(R) \neq 2$, we obtain

(3.18) $v[u, w],w]w + [(F(u)w + ud(w))vd(w), w] = 0$ for all $u, v, w \in$.

Putting $v = wv$ in (3:16) and using the fact that char$(R) \neq 2$, we get

(3.19) $w[v[u,w], w] + [F(u)wvd(w),w] = 0$ for all $u, v, w \in U$.

Subtracting (3:19) from (3:18), we have

(3.20) $[[v[u, w], w], w] + [ud(w)vd(w), w] = 0$ for all $u, v, w \in U$.

Replacing u by uw in (3:20), we obtain

(3.21) $[[v[u, w], w], w]w + [uwd(w)vd(w), w] = 0$ for all $u, v, w \in U$.

Right multiplying (3:20) by w and then subtracting from (3:21), we get

(3.22) $[u[d(w)vd(w), w];w] = 0$ for all $u, v, w \in U$.

Now, we substitute $d(w)vd(w)u$ for u in (3:22) and get

$$\begin{aligned} 0 &= [d(w)vd(w)u[d(w)vd(w), w],w] \\ &= d(w)vd(w)[u[d(w)vd(w), w], w] + \\ &\quad [d(w)vd(w), w]u[d(w)vd(w),w]: \end{aligned} \tag{3.23}$$

By using (3:22), it reduces to

(3.24) $[d(w)vd(w), w]u[d(w)vd(w), w] = 0$ for all $u, v, w \in U$.

Since U is a Lie ideal, it follows that

$[d(w)vd(w),w]u[d(w)vd(w), w] = (0)$ for all $v, w \in U$. Application of Lemma 2.1 yields that

(3.25) $[d(w)vd(w), w] = 0$ for all $v, w \in U$.

That is,

(3.26) $d(w)vd(w)w - wd(w)vd(w) = 0$ for all $v, w \in U$.

Now we put $v = 4vd(w)t$ and using the fact that char(R) $\neq 2$, we obtain

(3.27) $d(w)vd(w)td(w)w - wd(w)vd(w)td(w) = 0$ for all $t, v, w \in U$.

By (3:26), this can be written as

(3.28) $d(w)vwd(w)td(w) - d(w)vd(w)wtd(w) = 0$ for all $t, v, w \in U$.

That is,

(3.29) $d(w)v[d(w), w]td(w) = 0$ for all $t, v,w \in U$.

This implies $[d(w), w]v[d(w), w]t[d(w), w] = 0$ for all $t, v, w \in U$. Therefore, $[d(w), w]U[d(w), w]U[d(w), w] = (0)$ for all $w \in U$. Since R is prime, again by Lemma 2.1, we find that $[d(w), w] = 0$ for all $w \in U$. In view of Lemma 2.2, we conclude that $U \subseteq Z(R)$.

In the same manner the conclusion can be obtained in case $F(u)F(v) + vu \in Z(R)$ for all $u, v \in U$. Hence, the theorem is now proved.

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