A Two-Dimensional Problem in Generalized Thermoelasticity with Rotation

Nihar Sarkar

Purba Banbania Bhagabati Vidhyamandir, Habra, 24-PGS (N),
West Bengal, India
niharsarkar5@gmail.com

ABSTRACT

The present paper is aimed at studying the effect of rotation on the general model of the equations of generalized thermoelasticity for a homogeneous isotropic elastic half-space solid. The problem is in the context of Lord-Şhulman's (L-S) theory of generalized thermoelasticity as well as the classical dynamical coupled theory (CD). The normal mode analysis is used to obtain the exact expressions for the temperature, displacement components and the stresses distribution. The variations of the considered field variables through the horizontal distance are illustrated graphically and analyzed. Comparisons are made with the results in the presence and absence of rotation

1. INTRODUCTION

The coupled theory of thermoelasticity has been extended by including the thermal relaxation time in the constitutive equations by Lord and Shulman [1] and Green and Lindsay [2]. These theories eliminate the paradox of infinite velocity of heat propagation and are termed generalized theories of thermo-elasticity. Othman and Lotfy [3] studied two-dimensional problem of generalized magneto-thermoelasticity under the effect of temperature dependent properties. Othman and Lotfy [4] studied transient disturbance in a half-space under generalized magneto-thermoelasticity with moving internal heat source. Othman and Lotfy [5] studied the plane waves in generalized thermo-microstretch elastic half-space by using a general model of the equations of generalized thermo-microstretch for a homogeneous isotropic elastic half space. Othman and Lotfy [6] studied the generalized thermo-microstretch elastic medium with temperature dependent properties for different theories. Othman and Lotfy [7] studied the effect of magnetic field and inclined load in micropolar thermoelastic medium possessing cubic symmetry under three theories. The normal mode analysis was used to obtain the exact expression for the temperature distribution, thermal stresses, and the displacement components.

Agarwal [8, 9] studied respectively thermo-elastic and magneto- thermo-elastic plane wave propagation in an infinite non-rotating medium. Some problems in thermo- elastic rotating media are due to Schoenberg and Censor [10], Puri [11], Roy Choudhuri and Debnath [12,13] and Othman [14, 15]. Othman [16, 17] studied the effect of rotation in a micropolar generalized thermoelastic and thermo-viscoelasticity half space under different theories. The propagation of plane harmonic waves in a rotating elastic medium without thermal field has been studied. It was shown there that the rotation causes the elastic medium to be dispersive and an isotropic. These problems are based on more realistic elastic model since earth, moon and other planets have angular velocity

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2. FORMULATION OF THE PROBLEM

Following Lord and Şhulman [1], the constitutive equations and field equations for a linear isotropic generalized thermo-elastic solid in the absence of body forces are obtained, we consider rectangular coordinate system (x, y, z) having origin on the surface $y = 0$ and z-axis pointing vertically into the medium. The thermoelastic body is rotating uniformly with an angular velocity $\mathbf{\Omega} = \Omega \mathbf{n}$, where $\mathbf{n}$ is a unit vector representing the direction of the axis of rotation. The basic governing equations of linear generalized thermoelasticity with rotation in absence of body forces and heat sources are

$$(\lambda+\mu)\nabla(\nabla.\vec{u})+\mu\nabla^2\vec{u}-\gamma\nabla T=\rho[\ddot{\vec{u}}+\vec{\Omega}\times(\vec{\Omega}\times\vec{u})+2\vec{\Omega}\times\dot{\vec{u}}] \ , \quad (1)$$

$$k\nabla^2 T=\rho C_E\,(n_1+\tau_0\frac{\partial}{\partial t})\dot{T}+\gamma T_0\,(n_1+n_0\tau_0\frac{\partial}{\partial t})\dot{e} \quad (2)$$

$$\sigma_{ij}=2\mu e_{ij}+[\lambda u_{k,k}-\gamma(T-T_0)]\delta_{ij}, \quad (3)$$

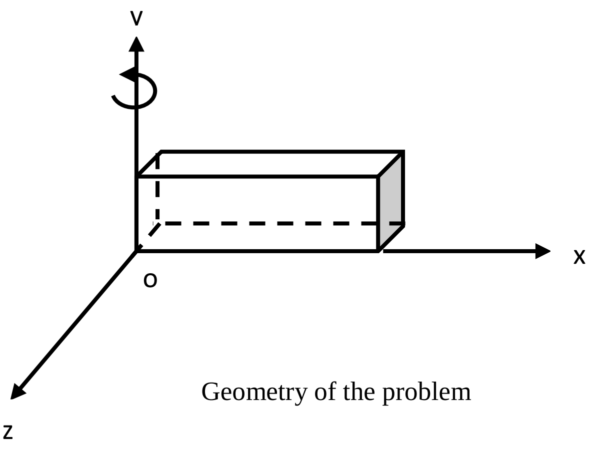
The state of plane strain parallel to the xz -plane is defined by

$$u_1=u(x,z,t)\ , \quad u_2=0\,, \quad u_3=w(x,z,t)\,, \ \Omega=(0,\Omega,0).$$

The field equations (1)-(3) reduce to

$$(\lambda+\mu)(u_{,xx}+w_{,xz})+\mu\nabla^2 u-\gamma T_{,x}=\rho[\ddot{u}-\Omega^2 u+2\Omega\dot{w}], \quad (4)$$

$$(\lambda+\mu)(u_{,xz}+w_{,zz})+\mu\nabla^2 w-\gamma T_{,z}=\rho[\ddot{w}-\Omega^2 w-2\Omega\dot{u}], \quad (5)$$



Geometry of the problem

$$k\nabla^2 T=\rho C_E\,(n_1+\tau_0\frac{\partial}{\partial t})\dot{T}+\gamma T_0\,(n_1+n_0\tau_0\frac{\partial}{\partial t})\dot{e}, \quad (6)$$

where $\gamma=(3\lambda+2\mu+k)\alpha_t$ and $\nabla^2=\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial z^2}$.

Equations (3)-(6) are the field equations of the generalized thermo- elastic solid, applicable to the L-S theory ($n_0=n_1=1,\ \ \tau_0>0$), and the CD theory ($n_0=0,\ \ n_1=1,\ \ \tau_0=0$).

For convenience, the following non-dimensional variables are used:

$$\bar{x}_i=\frac{\omega^*}{c_2}x_i,\ \bar{u}_i=\frac{\rho c_2\omega^*}{\gamma T_0}u_i,\ \bar{t}=\omega^* t,\ \bar{\tau}_0=\omega^*\tau_0,\ \bar{v}_0=\omega^* v_0,$$

$$\bar{T}=\frac{T-T_0}{T_0},\ \bar{\sigma}_{ij}=\frac{\sigma_{ij}}{\gamma T_0},\ \omega^*=\frac{\rho C_E c_2^2}{k},\ \bar{\Omega}=\frac{\Omega}{\omega^*},\ c_2^2=\frac{\mu}{\rho}. \quad (7)$$

Using the (7), Eqs. (4)-(6) become (dropping the dashed for convenience)

$$\ddot{u}-\Omega^2 u+2\Omega\dot{w}=\frac{\mu}{\rho c_2^2}\nabla^2 u+\frac{(\lambda+\mu)}{\rho c_2^2}e_{,x}-T_{,x}\,, \quad (8)$$

$$\ddot{w}-\Omega^2 w-2\Omega\dot{u}=\frac{\mu}{\rho c_2^2}\nabla^2 w+\frac{(\lambda+\mu)}{\rho c_2^2}e_{,z}-T_{,z}\,, \quad (9)$$

$$\nabla^2 T-(n_1+\tau_0\frac{\partial}{\partial t})\dot{T}=\frac{\gamma^2 T_0}{\rho k\omega^*}(n_1+n_0\tau_0\frac{\partial}{\partial t})\dot{e}\,. \quad (10)$$

Assuming the scalar potential functions $\varphi(x,z,t)$ and $\psi(x,z,t)$ defined by the relations in the non-dimensional form:

$$u=\frac{\partial\varphi}{\partial x}+\frac{\partial\psi}{\partial z},\ w=\frac{\partial\varphi}{\partial z}-\frac{\partial\psi}{\partial x},\ e=\nabla^2\varphi\,. \quad (11)$$

Using (11) in Eqs. (8)-(10), we obtain.

$$\left[\nabla^2-a_0\frac{\partial^2}{\partial t^2}+a_0\Omega^2\right]\varphi-a_0 T+2\Omega a_0\dot{\psi}=0 \quad (12)$$

$$\left[\nabla^2-a_1\frac{\partial^2}{\partial t^2}+a_1\Omega^2\right]\psi-2\Omega a_1\dot{\varphi}=0 \quad (13)$$

$$\left[\nabla^2-(n_1\frac{\partial}{\partial t}+\tau_0\frac{\partial^2}{\partial t^2})\right]T-=\varepsilon\,(n_1\frac{\partial}{\partial t}+n_0\tau_0\frac{\partial^2}{\partial t^2})\nabla^2\varphi, \quad (14)$$

where $c_1^2=\dfrac{\lambda+2\mu}{\rho}$, $a_0=\dfrac{c_2^2}{c_1^2}$, $a_1=\dfrac{\rho c_2^2}{\mu}$, $\varepsilon=\dfrac{\gamma^2 T_0}{\rho\omega^* k}$.

3. NORMAL MODE ANALYSIS

The solution of the considered physical variables can be decomposed in terms of normal mode as the following form:

$$[\varphi,\psi,\sigma_{ij},T](x,z,t)=[\bar{\varphi}(x),\bar{\psi}(x),\bar{\sigma}_{ij}(x),\bar{T}(x)]\exp(\omega t+iaz), \quad (15)$$

where $\bar{\varphi}(x)$ etc. are the amplitude of the function $\varphi(x)$, ω is a complex time constant and a is the wave number in the z-direction.

Using Eq. (15), then Eqs. (12) – (14) become respectively

$$(D^2-A_1)\bar{\varphi}-a_0\bar{T}+a_0A_2\bar{\psi}=0, \quad (16)$$

$$(D^2-A_3)\bar{\psi}-a_1A_2\bar{\varphi}=0, \quad (17)$$

$$(D^2-A_4)\bar{T}-A_5(D^2-a^2)\bar{\varphi}=0, \quad (18)$$

where $D=\frac{d}{dx}$, $A_1=a^2+a_0(\omega^2-\Omega^2)$,

$A_2=2\Omega\omega$, $A_3=a^2+a_1(\omega^2-\Omega^2)$,

$A_4=a^2+\omega\left(n_1+\tau_0\omega\right)$, $A_5=\varepsilon\omega\left(n_1+n_0\tau_0\omega\right)$.

Eliminating $\bar{\varphi},\bar{\psi},\bar{T}$ between Eqs. (16)-(18), we get the following sixth order ordinary differential equation satisfied by $\bar{\varphi},\bar{\psi},\bar{T}$

$$\left[D^6-\beta_1D^4+\beta_2D^2-\beta_3\right]\left\{\bar{\varphi}(x),\bar{\psi}(x),\bar{T}(x)\right\}=0. \quad (19)$$

Equation (19) can be factorized as

$$(D^2-k_1^2)(D^2-k_2^2)(D^2-k_3^2)\left\{\bar{\varphi}(x),\bar{\psi}(x),\bar{T}(x)\right\}=0. \quad (20)$$

where

$\beta_1=A_1+A_2+A_3+a_0A_5$,

$\beta_2=[A_1A_3+A_1A_4+A_3A_4+a_0a_1A_2^2+a_0A_5(a^2+A_3)$

$\beta_3=-(A_1A_2A_3+a_0a_1A_2^2A_4+a_0a^2A_3A_5)$.

The solution of Eq. (19), has the form

$$\bar{\varphi}=\sum_{n=1}^{3}M_n(a,\omega)\ e^{-k_nx}, \quad (21)$$

$$\bar{\psi}=\sum_{n=1}^{3}M'_n(a,\omega)\ e^{-k_nx}, \quad (22)$$

$$\bar{T}=\sum_{n=1}^{3}M''_n(a,\omega)\ e^{-k_nx}, \quad (23)$$

where $M_n(a,\omega)$, $M'_n(a,\omega)$, and $M''_n(a,\omega)$ are some parameters depending on a, ω and k_n^2 $(n=1,2,3)$ are the roots of the characteristic equation of Eq. (20).

Using Eqs. (21)-(23) into Eqs. (12) and (13) we get the following relations

$$\bar{\psi}=\sum_{n=1}^{3}g_nM_n(a,\omega)\ e^{-k_nx}, \quad (24)$$

$$\bar{T}=\sum_{n=1}^{3}h_nM_n(a,\omega)\ e^{-k_nx}, \quad (25)$$

where,

$$g_n=\frac{(k_n^2-a^2)}{\left[k_n^4-(a^2+A_3)k_n^2\right]},\quad h_n=\frac{A_5\left(k_n^2-a^2\right)}{k_n^2-A_4}.$$

Using Eqs. (21), (24) and (25) in the Eq. (10) and the non-dimensional form of Eq. (3), we obtain the expressions of displacement and stress components as follows:

$$\bar{u}=\sum_{n=1}^{3}(iag_n-k_n)M_n(a,\omega)\ e^{-k_nx}, \quad (26)$$

$$\bar{w}=\sum_{n=1}^{3}(ia-g_nk_n)M_n(a,\omega)\ e^{-k_nx}, \quad (27)$$

$$\bar{\sigma}_{zz}=\sum_{n=1}^{5}\delta_nM_n(a,\omega)\ e^{-k_nx}, \quad (28)$$

$$\bar{\sigma}_{xz}=\sum_{n=1}^{5}\xi_nM_n(a,\omega)\ e^{-k_nx}, \quad (29)$$

where

$\delta_n=iaf_1(ia-g_nk_n)-f_2k_n(iag_n-k_n)-h_n$,

$\xi_n=iaf_3(iag_n-k_n)-k_nf_4(ia-g_nk_n)$, $f_1=\frac{\lambda+2\mu}{\rho c_2^2}$,

$f_2=\frac{\lambda}{\rho c_2^2}$, $f_3=\frac{\mu}{\rho c_2^2}$, $f_4=\frac{\mu}{\rho c_2^2}$.

4. APPLICATION

The plane boundary subjects to an instantaneous normal point force and the boundary surface is isothermal, the boundary conditions x=0 are:

$$\mathrm{T}(0,\mathrm{z},\mathrm{t}) = 0,\ \sigma_{zz} = -p_0, \qquad \sigma_{xz} = 0, \tag{30}$$

Applying the boundary conditions (30) at the surface $x = 0$ of the medium, we obtain a system of three equations in the unknowns M_n $(n = 1, 2, 3)$. After applying the inverse of matrix method, we the solutions as:

$$\begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix} = \begin{pmatrix} h_1 & h_2 & h_3 \\ \delta_1 & \delta_2 & \delta_3 \\ \xi_1 & \xi_2 & \xi_3 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ -p \\ 0 \end{pmatrix} \tag{31}$$

5. NUMERICAL RESULTS

In order to illustrate our theoretical results obtained in preceding section and to compare these in the context of various theories of thermoelasticity, we now present some numerical results. In the calculation process, we take the case of magnesium crystal [18] as material subjected to mechanical and thermal disturbances. Since, ω is the complex constant, then we taken $\omega = \omega_0 + i\zeta$. The other constants of the problem are taken as $\omega_0 = -2$, $\zeta = 1$, $\tau_0 = 0.02$ and the physical constants used are:

$\rho = 1.74\ \mathrm{gm/cm^3}, \lambda = 9.4\times10^{11}\ \mathrm{dyne/cm^2}, \mu = 4.0\times10^{11}\ \mathrm{dyne/cm^2}, T_0 = 23\ ^{o}C,$
$k = 0.6\times10^{-2}\ \mathrm{cal/cm\ sec}\ ^{o}C,\ \gamma = 0.779\times10^{-4}\ \mathrm{dyne},\ C_E = 0.23\ \mathrm{cal/gm}\ ^{o}C.$

The variation of the temperature distribution T, components of displacement u and w, normal stress σ_{zz} and tangential stress σ_{xz} with distance x at the plane $z = 1$ and $p_0 = 2$ for CD and LS theories have been shown by solid and dashed lines respectively for generalized thermoelasticity medium with rotation ($\Omega = 0.2$) and without rotation ($\Omega = 0.0$). These distributions are shown graphically in Figs. 1-5 for thermal sources for time $t = 0.1$. We notice that the results for the temperature, the displacement and stress distributions when the relaxation time is including in the heat equation are distinctly different from those when the relaxation time is not mentioned in heat equation, because the thermal waves in the Fourier's theory of heat equation travel with an infinite speed of propagation as opposed to finite speed in the non-Fourier case. This demonstrates clearly the difference between the coupled and the generalized theories of thermoelasticity.

6. CONCLUSIONS

- The curves in the context of the CD and LS theories decrease exponentially with increasing x, this indicates that the thermoelastic waves are un-attenuated and non-dispersive, where purely thermoelastic waves undergo both attenuation and dispersion.
- The curves of the physical quantities with LS model in most of figures are lower in comparison with those under CD model.

(29)

- Analytical solutions based upon normal mode analysis for themoelastic problem in solids have been developed and utilized.
- It can be concluded that a change of volume is attended by a change of the temperature while the effect of the deformation upon the temperature distribution is the subject of the theory of thermoelasticity.
- The value of all the physical quantities converges to zero with an increase in distance x and all functions are continuous.
- The presence of rotation plays a significant role in all the physical quantities.

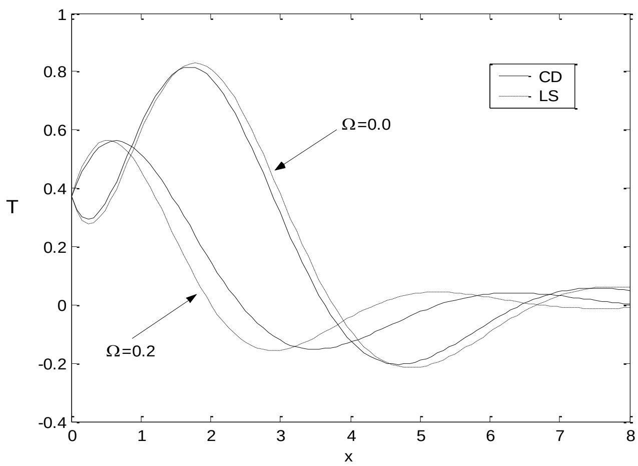
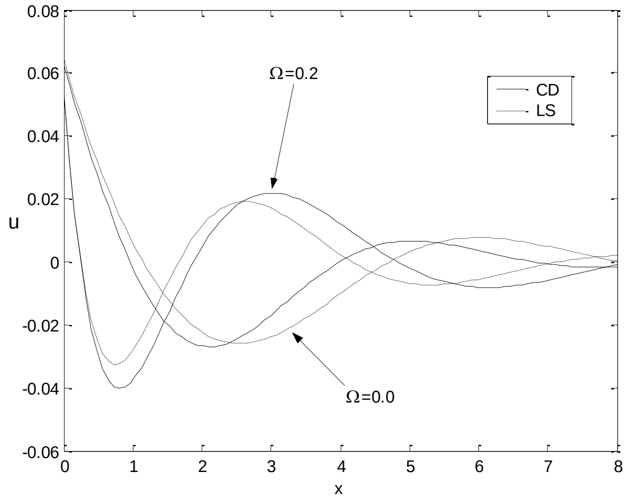
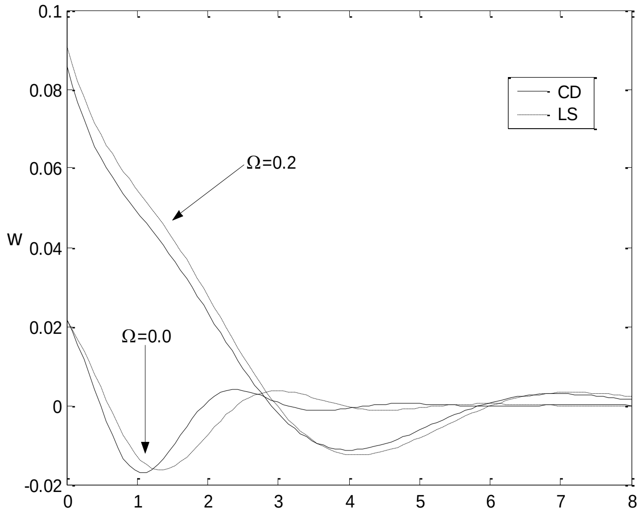


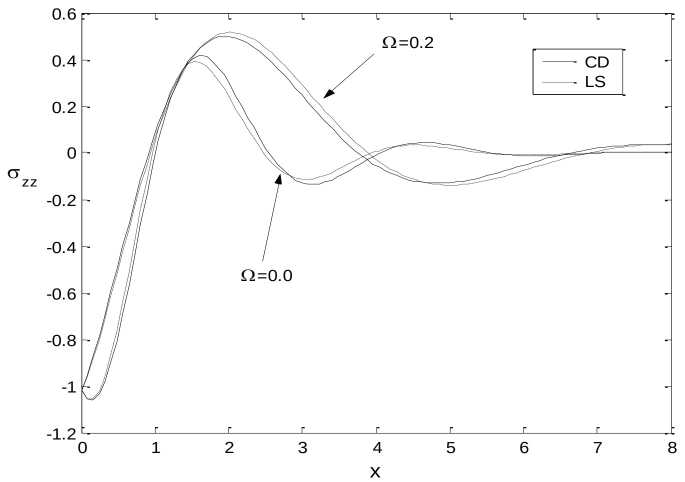
Fig 1- Temperature distribution at different rotation



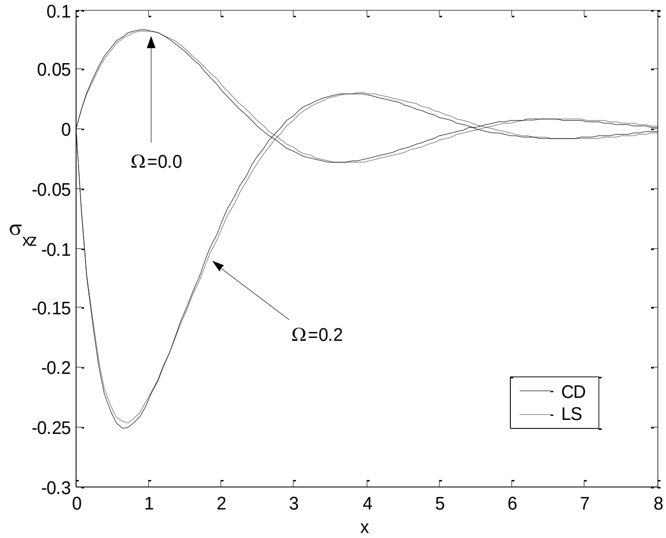
Fig—2: Normal displacement distribution at different rotation



Fig—3: Horizontal displacement distribution at different rotation



Fig—4: Normal stress distribution at different rotation



Fig—5: Tangential stress distribution at different rotation

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