Identities Related to Generalized Derivations on Lie Ideals

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ABSTRACT

Let R be an associative ring. An additive mapping F: R→R iscalled generalized derivations if there exists a derivation d : R → R such that $F(xy) = F(x)y + xd(y)$ for all $x,y \in R$. The objective of the present paper is to prove the following identities: (i) $F(x)x = G(x)x$, (ii) $[F(x),x] = [G(x),x]$, (iii) $F([x,y]) = [F(x),y] + [d(y),x]$ and $F(x \circ y) = (F(x) \circ y) + (d(y) \circ x)$ for all x,y in some appropriate subset of R.

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1. INTRODUCTION

Throughout the discussion, unless otherwise mentioned, R will denote an associative ring having at least two elements with center $Z(R)$. However, R may not have unity. For any $x,y \in R$, the symbol $[x,y]$ (resp. $(x \circ y)$) will denote the commutator $xy - yx$. For any $x,y \in R$; the symbol $[x,y]$ will denote the commutator $xy - yx$. The least positive integer n such that $nx = 0$ for all $x \in R$ is called the characteristic of the ring R and generally expressed as char(R). If no such positive integer exists, then R is said to have characteristic 0. Obviously, if char(R) $\neq$ m, then $mx = 0$ for some $x \in R$. Recall that a ring R is said to be prime if for any $a,b \in R$; $aRb = (0)$ implies $a = 0$ or $b = 0$. An additive subgroup U of R is said to be a Lie ideal of R if $[u,r] \in U$ for all $u \in U$ and $r \in R$. Let S be a non empty subset of R. A mapping $f:R \to R$ is called centralizing on S if $[f(x),x] \in Z(R)$ for all $x \in S$ and is called commuting on S if $[f(x),x] = 0$ for all $x \in S$.

An additive mapping $d:R\to R$ is said to be a derivation of R if $d(xy) = d(x)y + xd(y)$ for all $x,y \in R$. A derivation d is said to be inner if there exists $a \in R$ such that $d(x) = ax - xa$ for all $x \in R$: An additive mapping $F : R \to R$ is called a generalized derivation if there exists a derivation such that $F(xy) = F(x)y + xd(y)$ holds for all $x,y\in R$. Obviously, any derivation is a generalized derivation, but the converse is not true in general. A significant example is a map of the form $G(x) = ax+xb$ for some $a,b \in R$; such generalized derivations are called inner. Generalized derivations have been primarily studied on operator algebras. Therefore, any investigation from the algebraic point of view might be interesting (see for example [10] and [11]). In [11], Lee extended the definition of generalized derivation as follows: by a generalized derivation he means an additive mapping $G : J \to U$ such that $G(xy) = G(x)y + xd(y)$ for all $x,y \in J$, where U is the left Utumi quotient ring of R, J is a dense right ideal of R and d is a derivation from J to U. He also showed that every generalized derivation can be uniquely extended to a generalized derivation of U. In fact, there exists $a \in U$ and a derivation d of U such that $G(x) = ax + d(x)$ for all $x \in U$. Considerable work has been done on generalized derivations in prime and semiprime rings during the last few years (viz.; [5], [6], [7], [8], [9], [13], [14], [15], where further references can be found).

The purpose of this paper is to extend partially the results proved by Ashraf et al. [2] in the setting of Lie ideal of the prime ring R.

2. PRELIMINARIES RESULTS

Before stating our main result, let us list some basic facts, which will be used in the following:

Lemma 2.1. [3, Lemma 4] Let R be a prime ring with characteristic not two, $a,b \in R$. If U a non central Lie ideal of R and $aUb = (0)$, then $a = 0$ or $b = 0$.

Lemma 2.2. [12, Theorem 7] Let R be a prime ring with characteristic not two and U be a nonzero Lie ideal of R. If d is a nonzero derivation of R such that $[u,d(u)] \in Z(R)$, then $U \subseteq Z(R)$.

3. MAIN RESULTS

Theorem 3.1. Let R be a prime ring such that char(R) $\neq$2 and U be a Lie ideal of R such that $u^2 \in U$, for all $u \in U$. If there exists an additive mapping $F : R \to R$ associated with a derivation d of R such that $F(xy) = F(x)y + xd(y)$ for all $x,y \in R$. If $F(u)u = uG(u)$ for all $u \in U$ or $F(u)u+uG(u) = 0$ for all $u \in U$, then $U \subseteq Z(R)$.

Proof. First we consider the case

$$F(u)u = uG(u) \text{ for all } u \in U$$

Linearizing the above relation, we have

(3.1) $F(u)v + F(v)u = uG(v) + vG(u)$ for all $u,v \in U$.

Replacing u by $2uv$ in above equation, we get

(3.2) $2(F(u)v^2+ud(v)v+F(v)uv) = 2(uvG(v)+vG(u)v+vug(v))$ for all $u,v \in U$.

Since char(R) $\neq$2, we have

(3.3) $F(u)v^2 + ud(v)v + F(v)uv = uvG(v) + vG(u)v + vug(v)$ for all $u,v \in U$.

Right multiplication by v to the relation (3.1), we obtain

(3.4) $F(u)v^2 + F(v)uv = uG(v)v + vG(u)v$ for all $u,v \in U$.

Combination of (3.3) and (3.4) yield that

(3.5) $ud(v)v = vug(v) + u[v,G(v)]$ for all $u,v \in U$.

Now, replacing u by $2wu$ in (3.5), we get

(3.6) $2wud(v)v = 2(vwug(v) + wu[v,G(v)])$ for all $u,v,w \in U$.

Char(R) $\neq$2 implies that

(3.7) $wud(v)v = vwug(v) + wu[v,G(v)]$ for all $u,v,w \in U$.

Left multiplying to (3.5) by w, we obtain

(3.8) $wud(v)v = wvug(v) + wu[v,G(v)]$ for all $u,v,w \in U$.

Combining (3.7) and (3.8), we get $[v,w]ug(v) = 0$ for all $u,v,w \in U$ and hence $[v,g(v)]ug(v) = 0$ for all $u,v \in U$. That is, $[v,g(v)]U[v,g(v)] = (0)$ for all $u \in U$.

In view of Lemma 2.1, we find that $[v,g(v)] = 0$. Application of Lemma 2.2 yields

$U \subseteq Z(R)$.

Using the same approach as we have used above, we can get the required result

for $F(u)u + uG(u) = 0$ for all $u \in U$. This completes the proof.

Theorem 3.2. Let R be a prime ring such that char(R) $\neq$2 and U be a Lie ideal of R such that $u^2 \in U$, for all $u \in U$. If there exists an additive mapping F : $R \to R$ associated with a derivation d of R such that $F(xy) = F(x)y+xd(y)$ for all $x,y \in R$. If $[F(u), v] = [u,G(v)]$ for all $u,v \in U$ or $[F(u),v] + [u,G(v)] = 0$ for all $u,v \in U$, then $U \subseteq Z(R)$.

Proof. We begin with the situation

(3.9) $[F(u),v] = [u,G(v)]$ for all $u,v \in U$.

Replacing v by $2vu$ in the above relation, we get

(3.10) $2(v[F(u),u]) = 2([u,v]g(x) + v[u,g(u)]) \in Z(R)$ for all $u,v \in U$.

Since char(R) $\neq$2, we obtain

(3.11) $v[F(u),u] = [u,v]g(x) + v[u,g(u)] \in Z(R)$ for all $u,v \in U$.

Again replace v by wv in (3.11) and use it, to get $[u,w]vg(u) = 0$ for all $u, v,w \in U$.

Replacing w by $g(u)$, we obtain $[u,g(u)]vg(u) = 0$ for all $u,v,w \in U$. This further implies that $[u,g(u)]U[u,g(u)] = (0)$ for all $u \in U$. Application of Lemmas 2.1 and 2.2 yield $U \subseteq Z(U)$. This proves the theorem.

Theorem 3.3. Let R be a prime ring such that char(R) $\neq$2 and U be a Lie

ideal of R such that $u^2 \in U$, for all $u \in U$. If there exists an additive mapping $F : R \to R$ associated with a derivation d of R such that $F(xy) = F(x)y + xd(y)$ for all $x,y \in R$. If $F([u,v]) = [F(u),v] + [d(v),u]$ for all $u,v \in U$, then $U \subseteq Z(R)$.

Proof. In view of the hypothesis, we have

(3.12) $F([u,v]) = [F(u),v] + [d(v),u]$ for all $u,v \in U$.

Replacing v by $2vu$ in (3.12) and using (3.12), we find that

(3.13) $4[u,v]d(u) = 2(v[F(u),u] + v[d(u),u])$ for all $u,v \in U$.

Since char(R) $\neq$2, we obtain

(3.14) $2[u,v]d(u) = v[F(u),u] + v[d(u),u]$ for all $u,v \in U$.

Now replace v by vw in (3.14) and use (3.14), to get $2[u,v]wd(u) = 0$ for all $u,v,w \in U$. Using char(R) $\neq$2, we get $[u,v]wd(u) = 0$ for all $u,v,w \in U$.

Replacing v by $d(u)$ and proceeding as above we arrive at $[u,d(u)]U[u,d(u)] = (0)$ for all $u \in U$. Proceeding same as above theorems, we get the required result.

Theorem 3.4. Let R be a prime ring such that char(R) $\neq 2$ and U be a Lie ideal of R such that $u^2 \in U$, for all $u \in U$. If there exists an additive mapping $F : R \rightarrow R$ associated with a derivation d of R such that $F(xy) = F(x)y + xd(y)$ for all $x,y \in R$. If $F(u \circ v) = (F(u) \circ v)+(d(v) \circ u)$ for all $u,v \in U$, then $U \subseteq Z(R)$.

Proof. We have

(3.15) $F(u \circ v) = (F(u) \circ v) + (d(v) \circ u)$ for all $u,v \in U$.

Interchange v by $2vu$ in (3.15) and using it, we obtain

(3.16) $(u \circ v)d(u)) = 2(-v[F(u),v] - v(d(u) \circ u) + [v,u]d(u))$ for all $u,v \in U$.

Since char(R) $\neq 2$, we get

(3.17) $(u \circ v)d(u) = -v[F(u),v] - v(d(u) \circ u) + [v,u]d(u)$ for all $u,v \in U$.

Replacing v by wv in (3.17) and using (3.17), to get $2[u,w]vd(u) = 0$ for all $u,v,w \in U$. Char(R) $\neq 2$ implies that $[u,w]vd(u) = 0$ for all $u,v,w \in U$. Now by the same techniques as we used to prove Theorem 3.4, we get the required result. This completes the proof.

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