Cartesian Product on Fuzzy Prime and Fuzzy Semiprime Ideals of Ordered Semigroups

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ABSTRACT

In this paper, we give some results on cartesian product of fuzzy prime and fuzzy semiprime ideals of ordered semigroups.

Keywords — Ordered Semigroup, Fuzzy Ideal, Fuzzy Prime Ideal, Fuzzy Semiprime Ideal.

1. INTRODUCTION

The concept of a fuzzy set was introduced initially by Zadeh [9]. Rosenfeld [5] considered the concept of fuzzy subgroups. Kuroki [2] studied fuzzy ideals of semigroups. Kehayopulu and Tsingelis in [3] introduced fuzzy sets in ordered semigroups/ordered groupoids. They also studied fuzzy bi-ideals and fuzzy quasi-ideals in ordered semigroups [4]. In [1], Ersoy, Tepecik and Demir discussed cartesian product of fuzzy prime ideals of rings. Samit Kumar Majumder and Sujit Kumar Surdar [6] have studied cartesian product of fuzzy prime and fuzzy semiprime ideals of semigroups. In this paper, we study the cartesian product of fuzzy ideals, fuzzy prime ideals and fuzzy semiprime ideals of ordered semigroups and the characterizations of the above fuzzy ideals are also given.

2. PRELIMINARIES

Definition 2.1 By an ordered semigroup (po-semigroup), we mean an ordered set $(S, \leq)$ at the same time a semigroup satisfying the following conditions:

$a \leq b \Rightarrow xa \leq xb$ and $ax \leq bx \ \forall \ a, b, x \in S$.

Definition 2.2 Let $(S, ., \leq)$ be an ordered semigroup and $\phi \neq A \subseteq S$. A is called a left (resp. right) ideal of S if it satisfies:

- $SA \subseteq A$ (*resp.* $AS \subseteq A$).
- If $a \in A$, $S \ni b \leq a$, then $b \in A$.

A is called an ideal of S if it is both a left and a right ideal of S.

Definition 2.3 Let $(S, ., \leq)$ be an ordered semigroup and $\phi \neq T \subseteq S$. Then T is called prime if $xy \in T \Rightarrow x \in T$ or $y \in T$ for all $x, y \in S$.

Let T be an ideal of S. If T is prime subset of S, then T is called a prime ideal.

Definition 2.4 Let $(S, ., \leq)$ be an ordered semigroup and $\phi \neq T \subseteq S$. Then T is called semiprime if $a^2 \in T \Rightarrow a \in T$ for all $a \in S$.

Let T be an ideal of S. If T is semiprime subset of S, then T is called a semiprime ideal.

Definition 2.5 Let $(S, ., \leq)$ be an ordered semigroup. By a fuzzy subset μ of S, we mean a mapping $\mu: S \rightarrow [0, 1]$.

Definition 2.6 Let μ be any fuzzy subset of an ordered semigroup $(S, ., \leq)$. The set $\mu_t = \{x \in S / \mu(x) \geq t\}$, where $t \in [0, 1]$ is called a level subset of S.

Definition 2.7 Let $(S, ., \leq)$ be an ordered semigroup. A non-empty subset μ of S is called a fuzzy left (resp. right) ideal of S if

- $x \leq y \Rightarrow \mu(x) \geq \mu(y)$

- $\mu(xy) \geq \mu(y)$ (resp. $\mu(xy) \geq \mu(x)$) for all $x, y \in S$.

If μ is both a fuzzy left and a fuzzy right ideal of S, then μ is called a fuzzy ideal (fuzzy two-sided ideal) of S.

Equivalently, μ is called a fuzzy ideal of S if

- $x \leq y \Rightarrow \mu(x) \geq \mu(y)$
- $\mu(xy) \geq \max\{\mu(x), \mu(y)\}$ for all $x, y \in S$.

Definition 2.8 Let $(S, ., \leq)$ be an ordered semigroup. A non-empty fuzzy subset μ of S is called prime if $\mu(xy) = \max\{\mu(x), \mu(y)\}$ for all $x, y \in S$.

A fuzzy ideal μ of S is called a fuzzy prime ideal of S if μ is a prime fuzzy subset of S.

Definition 2.9 Let $(S, ., \leq)$ be an ordered semigroup. A non-empty fuzzy subset μ of S is called semiprime if $\mu(x) \geq \mu(x^2)$, for all $x \in S$.

A fuzzy ideal μ of S is called a fuzzy semiprime ideal of S if μ is a semiprime fuzzy subset of S.

Theorem 2.10 *[3]* Let $(S, ., \leq)$ be an ordered semigroup and μ be a fuzzy subset of S. Then μ is a fuzzy ideal of S if and only if for every $t \in [0, 1]$ $\mu_t (\neq \phi)$ is an ideal of S.

Theorem 2.11 *[8]* Let $(S, ., \leq)$ be an ordered semigroup and μ be a non-empty fuzzy subset of S. Then μ is a fuzzy prime ideal of S if and only if for every $t \in [0, 1]$ $\mu_t (\neq \phi)$ is a prime ideal of S.

Theorem 2.12 *[8]* Let $(S, ., \leq)$ be an ordered semigroup and μ be a non-empty fuzzy subset of S. Then μ is a fuzzy semiprime ideal of S if and only if for every $t \in [0, 1]$ $\mu_t (\neq \phi)$ is a semiprime ideal of S.

3. CARTESIAN PRODUCT OF FUZZY PRIME AND FUZZY SEMIPRIME IDEALS

Result 3.1 *[7]* Let $(S_1, ., \leq)$ and $(S_2, ., \leq)$ be two ordered semigroups. Under the co-ordinatewise multiplication, the cartesian product $S_1 \times S_2$ of S_1 and S_2 forms a semigroup. Define a partial order $\leq$ on $S_1 \times S_2$ by $(a, b) \leq (c, d)$ if and only if $a \leq_{S1} c$ and $b \leq_{S2} d$ for all $(a, b), (c, d) \in S_1 \times S_2$. Then $S_1 \times S_2$ is an ordered semigroup.

Definition 3.2 *[1]* Let μ and σ be two fuzzy subsets of X. Then the cartesian product of μ and σ is defined by $(\mu \times \sigma)(x, y) = \min\{\mu(x), \sigma(x)\}$ for all $x, y \in X$.

Definition 3.3 *[6]* Let μ and σ be two fuzzy subsets of X and $t \in [0, 1]$. Then $(\mu \times \sigma)_t = \mu_t \times \sigma_t$.

Proposition 3.4 Let μ and σ be two fuzzy left ideals (fuzzy right ideals, fuzzy ideals) of an ordered semigroup S. Then $\mu \times \sigma$ is a fuzzy left ideal (resp. fuzzy right ideal, fuzzy ideal) of an ordered semigroup $S \times S$.

Proof. Let μ and σ be two fuzzy left ideals of an ordered semigroup S and $(a, b), (c, d) \in S \times S$. If $(a, b) \leq (c, d)$, then $(\mu \times \sigma)(a, b) = \min\{\mu(a), \sigma(b)\} \geq \min\{\mu(c), \sigma(d)\}$ (since μ and σ are fuzzy left ideals of S and $a \leq c$, $b \leq d$) = $(\mu \times \sigma)(c, d)$, which implies that $(\mu \times \sigma)(a, b) \geq (\mu \times \sigma)(c, d)$. Now $(\mu \times \sigma)\{(a, b)(c, d)\} = (\mu \times \sigma)(ac, bd) = \min\{\mu(ac), \sigma(bd)\} \geq \min\{\mu(c), \sigma(d)\}$ (since μ and σ are fuzzy left ideals of S) = $(\mu \times \sigma)(c, d)$, which implies that $(\mu \times \sigma)\{(a, b)(c, d)\} \geq (\mu \times \sigma)(c, d)$. Hence $\mu \times \sigma$ is a fuzzy left ideal of $S \times S$. In a similar way, we can prove the other cases also.

Proposition 3.5 Let μ and σ be two fuzzy prime ideals of an ordered semigroup S. Then $\mu \times \sigma$ is a fuzzy prime ideal of an ordered semigroup $S \times S$.

Proof. Let μ and σ be two fuzzy prime ideals of an ordered semigroup S. Then μ and σ are fuzzy ideals of S. By Proposition 3.4, $\mu \times \sigma$ is a fuzzy ideal of $S \times S$. We need to show that $(\mu \times \sigma)\{(a, b)(c, d)\} = \max\{(\mu \times \sigma)(a, b), (\mu \times \sigma)(c, d)\}$ for all $(a, b), (c, d) \in S \times S$. Let $(a, b), (c, d) \in S \times S$. Then $(\mu \times \sigma)\{(a, b)(c, d)\} = (\mu \times \sigma)(ac, bd) = \min\{\mu(ac), \sigma(bd)\} = \min\{\max\{\mu(a), \mu(c)\}, \max\{\sigma(b), \sigma(d)\}\}$ (since μ and σ are fuzzy prime ideals of S) = $\max\{\min\{\mu(a), \sigma(b)\}, \min\{\mu(c), \sigma(d)\}\} = \max\{(\mu \times \sigma))(a, b), (\mu \times \sigma))(c, d)\}$. Thus $(\mu \times \sigma)\{(a, b)(c, d)\} = \max\{(\mu \times \sigma)(a, b), (\mu \times \sigma)(c, d)\}$. It follows that $\mu \times \sigma$ is prime.

Proposition 3.6 Let μ and σ be two fuzzy semiprime ideals of an ordered semigroup S. Then $\mu \times \sigma$ is a fuzzy semiprime ideal of an ordered semigroup $S \times S$.

Proof. Let μ and σ be two fuzzy semiprime ideals of an ordered semigroup S. Then μ and σ are fuzzy ideals of S. By Proposition 3.4, $\mu\times\sigma$ is a fuzzy ideal of $S\times S$. It is enough to show that for all $(a, b)\in S\times S$, $\mu\times\sigma$ is semiprime. Let $(a, b)\in S\times S$. Then $(\mu\times\sigma)(a, b) = \min\{\mu(a), \sigma(b)\} \geq \min\{\mu(a^2), \sigma(b^2)\}$ (since μ and σ are fuzzy semiprime ideals of S) $= (\mu\times\sigma)(a^2, b^2) = (\mu\times\sigma)(a, b)^2$, which implies that $(\mu\times\sigma)(a, b) \geq (\mu\times\sigma)(a, b)^2$. Hence $\mu\times\sigma$ is a fuzzy semiprime ideal of $S\times S$.

Proposition 3.7 Let S be an an ordered semigroup and let μ and σ be two fuzzy ideals of S. Then $\mu\times\sigma$ is a fuzzy ideal of $S\times S$ if and only if for all $t\in[0, 1]$ $(\mu\times\sigma)_t(\neq\phi)$ is an ideal of $S\times S$.

Proof. Let $\mu\times\sigma$ be a fuzzy ideal of $S\times S$. For any $t\in[0, 1]$ with $(\mu\times\sigma)_t(\neq\phi)$, we must show that the level sets $(\mu\times\sigma)_t$ are ideals of $S\times S$. i). Let $(a, b)\in(\mu\times\sigma)_t$ and $(c, d)\in S\times S$. Then $(\mu\times\sigma)(a, b)\geq t$. Since $\mu\times\sigma$ is a fuzzy ideal of $S\times S$, we have $(\mu\times\sigma)\{(a, b)(c, d)\} \geq \max\{(\mu\times\sigma)(a, b), (\mu\times\sigma)(c, d)\} \geq (\mu\times\sigma)(a, b) \geq t$, which implies that $(\mu\times\sigma)\{(a, b)(c, d)\}\geq t$. Thus $(a, b)(c, d)\in(\mu\times\sigma)_t$. Similarly we can prove $(c, d)(a, b)\in(\mu\times\sigma)_t$. ii). Let $(a, b)\in(\mu\times\sigma)_t$, $S\times S \ni (c, d)\leq(a, b)$. Then $(\mu\times\sigma)(a, b)\geq t$. Since $\mu\times\sigma$ is a fuzzy ideal of $S\times S$ and $(c, d)\leq(a, b)$, we have $(\mu\times\sigma)(c, d)\geq(\mu\times\sigma)(a, b)\geq t$. That is $(c, d)\in(\mu\times\sigma)_t$. Therefore $(\mu\times\sigma)_t$ is an ideal of $S\times S$.

Conversely, assume that for all $t\in[0, 1]$ such that $(\mu\times\sigma)_t\neq\phi$, the set $(\mu\times\sigma)_t$ is an ideal of $S\times S$. We need to prove that $\mu\times\sigma$ is a fuzzy ideal of $S\times S$. Let $(a, b), (c, d)\in S\times S$. i). If $(a, b)\leq(c, d)$, then $(\mu\times\sigma)(a, b)\leq(\mu\times\sigma)(c, d)$. In fact: Let $t_1 = (\mu\times\sigma)(c, d)$. Then $(c, d)\in(\mu\times\sigma)_{t1}$. Since $(\mu\times\sigma)_{t1}$ is an ideal of $S\times S$, $(c, d)\in(\mu\times\sigma)_{t1}$, $(a, b)\in S\times S$ and $(a, b)\leq(c, d)$, we have $(a, b)\in(\mu\times\sigma)_{t1}$. Then $(\mu\times\sigma)(a, b) \geq t_1 = (\mu\times\sigma)(c, d)$. Therefore condition i) is true. Now we have to prove the second condition, that is, $(\mu\times\sigma)\{(a, b)(c, d)\}\geq\max\{(\mu\times\sigma)(a, b), (\mu\times\sigma)(c, d)\}$. If $(a, b), (c, d)\in S\times S$, then $(\mu\times\sigma)\{(a, b)(c, d)\}\geq\max\{(\mu\times\sigma)(a, b), (\mu\times\sigma)(c, d)\}$. In fact, if $(\mu\times\sigma)\{(a, b)(c, d)\} < \max\{(\mu\times\sigma)(a, b), (\mu\times\sigma)(c, d)\}$, then there exists $t\in[0,1]$ such that $(\mu\times\sigma)\{(a, b)(c, d)\} < t < \max\{(\mu\times\sigma)(a, b), (\mu\times\sigma)(c, d)\}$ and so $(\mu\times\sigma)(a, b) > t$ or $(\mu\times\sigma)(c, d) > t$. Then $(a, b)\in(\mu\times\sigma)_t$ or $(c, d)\in(\mu\times\sigma)_t$. By hypothesis, $(a, b)(c, d)\in(\mu\times\sigma)_t$. Then $(\mu\times\sigma)\{(a, b)(c, d)\}\geq t$, impossible. It follows that $(\mu\times\sigma)\{(a, b)(c, d)\}\geq\max\{(\mu\times\sigma)(a, b), (\mu\times\sigma)(c, d)\}$. Therefore, $\mu\times\sigma$ is a fuzzy ideal of $S\times S$.

Proposition 3.8 Let μ and σ be two fuzzy prime ideals of an ordered semigroup S. Then the level subset $(\mu\times\sigma)_t$, $t\in[0, 1]$ is a prime ideal of an ordered semigroup $S\times S$.

Proof. Let μ and σ be two fuzzy prime ideals of an ordered semigroup S. Then by Proposition 3.5, $\mu\times\sigma$ is a fuzzy prime ideal of an ordered semigroup $S\times S$. This implies that $\mu\times\sigma$ is a fuzzy ideal of $S\times S$. Hence by Theorem 3.7, for all $t\in[0, 1]$ $(\mu\times\sigma)_t$ $(\neq\phi)$ is an ideal of $S\times S$. So we must show that the level ideals are prime. Suppose that $(a, b), (c, d)\in S\times S$ is such that $(a, b)(c, d)\in(\mu\times\sigma)_t$. Then $(\mu\times\sigma)\{(a, b)(c, d)\}\geq t$. Since $\mu\times\sigma$ is a fuzzy prime ideal of $S\times S$, we have $\max\{(\mu\times\sigma)(a, b), (\mu\times\sigma)(c, d)\} = (\mu\times\sigma)\{(a, b)(c, d)\}\geq t$. Then $(\mu\times\sigma)(a, b)\geq t$ or $(\mu\times\sigma)(c, d)\geq t$, that is, $(a, b)\in(\mu\times\sigma)_t$ or $(c, d)\in(\mu\times\sigma)_t$. Therefore, $(\mu\times\sigma)_t$ is prime.

Proposition 3.9 If the level subset $(\mu\times\sigma)_t$, $t\in[0, 1]$ of $\mu\times\sigma$ is a prime ideal of an ordered semigroup $S\times S$, then $\mu\times\sigma$ is a fuzzy prime ideal of an ordered semigroup $S\times S$.

Proof. Let $(\mu\times\sigma)_t$ $(\neq\phi)$ be a prime ideal of an ordered semigroup $S\times S$ for any $t\in[0, 1]$. Then by Theorem 3.7 and hypothesis, $\mu\times\sigma$ is a fuzzy ideal of $S\times S$. So we need to prove that $\mu\times\sigma$ is prime. Let $(a, b), (c, d)\in S\times S$ and $(\mu\times\sigma)\{(a, b)(c, d)\} = t$. Since $(\mu\times\sigma)_t$ $(\neq\phi)$ is a prime ideal of $S\times S$ and $(a, b)(c, d)\in(\mu\times\sigma)_t$, we have $(a, b)\in(\mu\times\sigma)_t$ or $(c, d)\in(\mu\times\sigma)_t$, which implies that $(\mu\times\sigma)(a, b)\geq t$ or $(\mu\times\sigma)(c, d)\geq t$. Then $\max\{(\mu\times\sigma)(a, b), (\mu\times\sigma)(c, d)\}\geq t = (\mu\times\sigma)\{(a, b)(c, d)\}\geq\max\{(\mu\times\sigma)(a, b), (\mu\times\sigma)(c, d)\}$. Thus $(\mu\times\sigma)\{(a, b)(c, d)\} = \max\{(\mu\times\sigma)(a, b), (\mu\times\sigma)(c, d)\}$. It follows that $\mu\times\sigma$ is prime.

Proposition 3.10 Let μ and σ be two fuzzy semiprime ideals of an ordered semigroup S. Then the level subset $(\mu\times\sigma)_t$, $t\in[0, 1]$ is a semiprime ideal of an ordered semigroup $S\times S$.

Proof. Let μ and σ be two fuzzy semiprime ideals of an ordered semigroup S. Then by Proposition 3.6, $\mu\times\sigma$ is a fuzzy semiprime ideal of an ordered semigroup $S\times S$. This implies that $\mu\times\sigma$ is a fuzzy ideal of $S\times S$. Hence by Theorem 3.7, for all $t\in[0, 1]$ $(\mu\times\sigma)_t$ $(\neq\phi)$ is an ideal of $S\times S$. So it is enough to show that the level ideals are semiprime. Let $(a, b)\in S\times S$

such that $(a, b)^2 \in (\mu \times \sigma)_t$. Then $(\mu \times \sigma)\{(a, b)^2\} \geq t$. Since $\mu \times \sigma$ is a fuzzy semiprime ideal of $S \times S$, we have $(\mu \times \sigma)(a, b) \geq (\mu \times \sigma)\{(a, b)^2\} \geq t$. Then $(a, b) \in (\mu \times \sigma)_t$. Thus, $(\mu \times \sigma)_t$ is semiprime.

Proposition 3.11 If the level subset $(\mu \times \sigma)_t$, $t \in [0, 1]$ of $\mu \times \sigma$ is a semiprime ideal of an ordered semigroup $S \times S$, then $\mu \times \sigma$ is a fuzzy semiprime ideal of an ordered semigroup $S \times S$.

Proof. Let $(\mu \times \sigma)_t$ $(\neq \phi)$ be a semiprime ideal of an ordered semigroup $S \times S$ for any $t \in [0, 1]$. Then by Theorem 3.7 and hypothesis, $\mu \times \sigma$ is a fuzzy ideal of $S \times S$. So we need to prove that $\mu \times \sigma$ is semiprime. Let $(a, b) \in S \times S$ and $(\mu \times \sigma)\{(a, b)^2\} = t$. Since $(\mu \times \sigma)_t$ $(\neq \phi)$ is a semiprime ideal of $S \times S$ and $(a, b)^2 \in (\mu \times \sigma)_t$, we have $(a, b) \in (\mu \times \sigma)_t$, which implies that $(\mu \times \sigma)(a, b) \geq t = (\mu \times \sigma)\{(a, b)^2\}$. Then $\mu \times \sigma$ is semiprime.

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